Chapter : 3

Ordinary differential equations: Euler's method, Taylor series method, Range Kutta II and IV order methods.

Differential equation is an equation involving derivatives of a function or functions. In other words, a differential equation contains one or more terms involving derivatives of one variable (the dependent variable, *y*) with respect to another variable (the independent variable, *x*).

For example, $\frac{dy}{dx} = 2x$

Differential equation represents the relationship between a continuously varying quantity and its rate of change. This is very essential in all scientific investigation.

Classification: Differential equations are classified into several broad categories, and these are in turn further divided into many subcategories. The most important categories are

Ordinary differential equations: When the function involved in the equation depends on only a single variable, its derivatives are ordinary derivatives and the differential equation is classed as an ordinary differential equation.

Definition:

In mathematics, an **ordinary differential equation** (**ODE**) is a **differential equation** containing one or more functions of one independent variable and its derivatives. The term **ordinary** is used in contrast with the term partial **differential equation** which may be with respect to more than one independent variable.

A differential equation that involves a function of a single variable and some of its derivatives is known as ordinary differential equation.

For example, $\frac{dy}{dx} = 2x^2 + 3x + 5$

Partial differential equations: On the other hand, if the function depends on several independent variables, so that its derivatives are partial derivatives, the differential equation is classed as a partial differential equation. The following are examples of ordinary differential equations:

$$\frac{dy}{dt} = -ky,$$

$$m\frac{d^2y}{dt^2} = -k^2y,$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0.$$

In these, *y* stands for the function, and either *t* or *x* is the independent variable. The symbols *k* and *m* are used here to stand for specific constants.

Whichever the type may be, a differential equation is said to be of the *n*th order if it involves a derivative of the *n*th order but no derivative of an order higher than this. The equation

$$\frac{\partial u}{\partial t} = k^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

is an example of a partial differential equation of the second order. The theories of ordinary and partial differential equations are markedly different, and for this reason the two categories are treated separately.

Instead of a single differential equation, the object of study may be a simultaneous system of such equations. The formulation of the laws of dynamics frequently leads to such systems. In many cases, a single differential equation of the *n*th order is advantageously replaceable by a system of *n* simultaneous equations, each of which is of the first order, so that techniques from linear algebra can be applied.

An ordinary differential equation in which, for example, the function and the independent variable are denoted by *y* and *x* is in effect an implicit summary of the essential characteristics of *y* as a function of *x*. These characteristics would presumably be more accessible to analysis if an explicit formula for *y* could be produced. Such a formula, or at least an equation in *x* and *y* (involving no derivatives) that is deducible from the differential equation, is called a solution of the differential equation. The process of deducing a solution from the equation by the applications of algebra and calculus is called solving or integrating the equation. It should be noted, however, that the differential equations that can be explicitly solved form but a small minority. Thus, most functions must be studied by indirect methods. Even its existence must be proved when there is no possibility of producing it for inspection. In practice, methods from numerical analysis, involving computers, are employed to obtain useful approximate solutions.

Order of a differential equation:

The order of a differential equation is the order of the highest derivative that appears in the equation. The above examples are both *first order* differential equations.

An example of a *second order* differential equation is $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$.

A number of differential equation cannot be solved by analytical method, it is therefore imperative to solve them by numerical methods. The different types of numerical methods to solve differential equations are:

1. Taylor’s series method
2. Runge kutta method
3. Euler’s method
4. Euler’s modified method
5. Picard method

Solution of a differential equation: Unlike algebraic equations, the solutions of differential equations **are functions and not just numbers**.

When a differential equation is solved using graphical method, the solution is a curve g(x,y) in the (x,y) the plane whose slope at every point (x,y) in the specified region is given by the equation dy/dx = f(x,y). The initial point (x,y) of the solution curve and the slope of the curve at this point are given with this information, we are asked to extrapolate the values of y for the set of values of x in the large (x1,xf). The fundamental method used in obtaining the solution of a differential equation is thus judicious extrapolation.

Euler's method

Euler's method is considered to be one of the oldest and simplest methods to find the numerical solution of ordinary differential equation or the initial value problems. Here, a short and simple algorithm and flowchart for Euler's method has been presented, which can be used to write program for the method in any high level programming language.

Through Euler's method, you can find a clear expression for y in terms of a finite number of elementary functions represented with x. The initial values of y and x are known, and for these an ordinary differential equation is considered.

Now, lets look at the mathematics and algorithm behind the Euler's method. A sequence of short lines is approximated to find the curve of solution; this means considering tangent line in each interval. Using the information obtained from here, the value of 'y_n' corresponding to the value of 'x_n' is to determined by dividing the length ($x_n - x$) into n intervals or strips.

So, strip width= $(x_n - x)/n$ and $x_n = x_0 + nh$.

Again, if m be the slope of the curve at point, $y_{1=} y_{0\,+\,m}(x_0\,, y_o)h$.

Now, from this all the intermediate 'y' values can be found. This method was developed by Leonhard Euler.

Formula:

$$Y_{i+1} = Y_i + h\,f(X_i, Y_i)$$

$$X_{i+1} = X_i + h$$

Advantages: **Euler's Method** is simple and direct can be used for nonlinear IVPs
Disadvantages: it is less accurate and numerically unstable. Approximation error is proportional to the step size h. Hence, good approximation is obtained with a very small h.

Euler's Method Algorithm:

1. Start
2. Define function
3. Get the values of x0, y0, h and xn
   *Here x0 and y0 are the initial conditions
   h is the interval
   xn is the required value

4. n = (xn – x0)/h + 1
5. Start loop from i=1 to n
6. y = y0 + h*f(x0,y0)
   x = x + h
7. Print values of y0 and x0
8. Check if x < xn
   If yes, assign x0 = x and y0 = y
   If no, goto 9.
9. End loop i
10. Stop

Note: Euler's method cannot be regarded as one of the best approaches to solve ordinary differential equations. It is very slow, but its modified form, Modified Euler's Method is fast.

Taylor Series method:

A **Taylor series** is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The concept of a Taylor series was formulated by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715.

A function can be approximated by using a finite number of terms of its Taylor series. Taylor's theorem gives quantitative estimates on the error introduced by the use of such an approximation. The polynomial formed by taking some initial terms of the Taylor series is called a Taylor polynomial. The Taylor series of a function is the limit of that function's Taylor polynomials as the degree increases, provided that the limit exists. A function may not be equal to its Taylor series, even if its Taylor series converges at every point. A function that is equal to its Taylor series in an open interval (or a disc in the complex plane) is known as an analytic function in that interval.

Advantages: One step, explicit; can be high order; convergence proof easy

Disadvantages: Needs the explicit form of f and of derivatives of f.

Formula:

$$Y(x) = Y(Xo) + \frac{X-X0}{1!}(Y`(Xo)) + \frac{X-Xo^2}{2!}(Y``(Xo) + \frac{X-Xo^3}{3!}(Y```(Xo)) + \frac{X-Xo^4}{4!} + (Y````(Xo)) + \ldots\ldots \frac{X-Xo^n}{n!} + Y^n(Xo)$$

Taylor series algorithm:

Specify **x_0, x_n, y_0, h**

(**(x_0, y_0)** Initial point,

x_n point where the solution is required

h the step length to be used in the marching process)

Repeat

compute **$f(x_i, y_i)$, $f'(x_i, y_i)$, $f''(x_i, y_i)$ . . .**

compute **$y(x_i+h) = y(x_i) + h\, f(x_i, y_i) + h^2/2\, f'(x_i, y_i) + h^3/3!\, f''(x_i, y_i) + \ldots$**

$\mathbf{x_i = x_i + h}$
until $\mathbf{x_i = x_n}$

Runge Kutta Methods: the **Runge–Kutta methods** are a family of implicit and explicit iterative methods, which includes the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations. These methods were developed around 1900 by the German mathematicians C. Runge and M. W. Kutta.

To avoid the disadvantage of the Taylor series method, we can use Runge-Kutta methods. These are still one step methods, but they depend on estimates of the solution at different points.
Advantages of Runge Kutta Methods:

1. One step method – global error is of the same order as local error.
2. Don’t need to know derivatives of f.
3. Easy for ”Automatic Error Control” . The length of the step can be modified at any time in t h e course of the computation without additional labour.
4. No special devices are required for starting the computation

On the other hand it is open to two major objections/ disadvantages:

(1) The process does not contain in itself any simple means for estimating the error or for detecting computation mistakes.
(2) Each step requires form substitutions into the differential equation

In general for an r^{th} order Runge-Kutta method we need S(r) evaluations of (f) for each timestep.
Second order Runge Kutta method/midpoint/heun’s method: The Runge-Kutta 2nd order method is a numerical technique used to solve an ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Only first order ordinary differential equations can be solved by using the Runge-Kutta 2nd order method. The second order method requires 2 evaluations of (f) at every timestep.

Formula:

$\mathbf{K_1 = f(X_i, Y_i)}$
$\mathbf{K_2 = f(X_i + h, \ Y_i + K_1 h)}$
$\mathbf{Y_{i+1} = Y_i + \frac{h}{2}\ (K_1 + K_2)}$

Fourth order Runge Kutta method.
The fourth order method requires 4 evaluations of (f) at every timestep.

Formula:

$\mathbf{K_1 = f(X_i, Y_i)}$
$\mathbf{K_2 = f(X_i + ½h, \ Y_i + ½\, K_1 h)}$
$\mathbf{K_3 = f(X_i + ½h, \ Y_i + ½\, K_2 h)}$

$$K_4 = f(X_i + h,\ Y_i + K_3 h)$$

$$Y_{i+1} = Y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)\,h$$