Equation

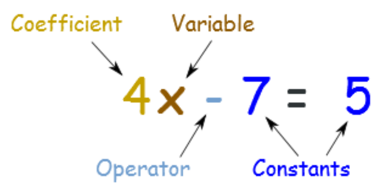
An equation says that two things are equal. It will have an equals sign "="

$$x + 2 = 6$$

That equation says: **what is on the left (x + 2) is equal to what is on the right (6)**

So an equation is like a **statement** "*this* equals *that*". An equation is an important part of calculus, which comes under mathematics. It helps us to solve many problems.

Parts of an Equation



A **Variable** is a symbol for a number we don't know yet. It is usually a letter like x or y.

A number on its own is called a **Constant**.

A **Coefficient** is a number used to multiply a variable (**4x** means **4** times **x**, so **4** is a coefficient)

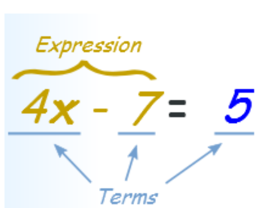
Variables without a number have a coefficient of 1 (**x** is really **1x**)

Sometimes a letter stands in for the number:

Example: $ax^2 + bx + c$

- **x** is a variable
- **a** and **b** are coefficients
- **c** is a constant

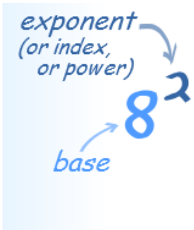
An **Operator** is a symbol (such as +, ×, etc) that shows an operation (ie we want to do something with the values).



A **Term** is either a single number or a variable, or numbers and variables multiplied together.

An **Expression** is a group of terms (the terms are separated by + or − signs).

The **exponent (positive)** (such as the 2 in x^2) says **how many times** to use the value in a ***multiplication.***



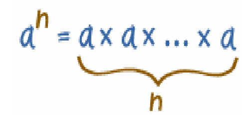
Examples:

$8^2 = 8 \times 8 = 64$

$y^3 = y \times y \times y$

$y^2z = y \times y \times z$

a^n tells you to multiply **a** by itself,
so there are **n** of those **a**'s:



Sometimes we use the ^ symbol (above the 6 on your keyboard), as it is easy to type.

Example: **2^4 is the same as 2^4**

- **2^4 = 2 × 2 × 2 × 2 = 16**

Negative Exponents

A negative exponent means how many times **to divide** one by the number.

Example: $8^{-1} = 1 \div 8 = 0.125$

You can have many divides:

Example: $5^{-3} = 1 \div 5 \div 5 \div 5 = 0.008$

This can be done in an easier way:

5^{-3} could also be calculated like:

$$1 \div (5 \times 5 \times 5) = 1/5^3 = 1/125 = 0.008$$

Examples:

$$a^{-n} = \frac{1}{a^n}$$

- Calculate the positive exponent (a^n)
- Then take the Reciprocal (i.e. $1/a^n$)

More Examples:

Negative Exponent		Reciprocal of Positive Exponent		Answer
4^{-2}	=	$1 / 4^2$	=	1/16 = 0.0625
10^{-3}	=	$1 / 10^3$	=	1/1,000 = 0.001
$(-2)^{-3}$	=	$1 / (-2)^3$	=	1/(-8) = -0.125

If the exponent is 1, then you just have the number itself (example $\mathbf{9^1}$ = **9**)
If the exponent is 0, then you get **1** (example $\mathbf{9^0}$ = **1**)
But what about $\mathbf{0^0}$? It could be either 1 or 0, and so it is considered as *"indeterminate"*.

Grouping

Use parenthesis to avoid confusions while grouping

With () :	$(-2)^2 = (-2) \times (-2) = 4$
Without () :	$-2^2 = -(2^2) = -(2 \times 2) = -4$

With () :	$(ab)^2 = ab \times ab$
Without () :	$ab^2 = a \times (b)^2 = a \times b \times b$

Fractional Exponents

An exponent of $^1/_2$ is actually **square root**	$4^{\frac{1}{2}} = \sqrt{4}$
An exponent of $^1/_3$ is **cube root**	$4^{\frac{1}{3}} = \sqrt[3]{4}$
An exponent of $^1/_4$ is **4th root**	$4^{\frac{1}{4}} = \sqrt[4]{4}$
	etc...

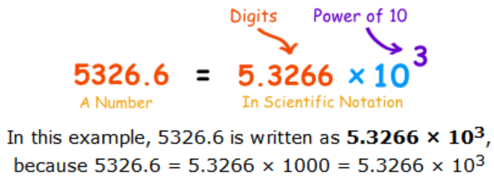
Laws of Exponents

Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = 1/x$	$4^{-1} = 1/4$
$x^m x^n = x^{m+n}$	$x^2x^3 = x^{2+3} = x^5$
$x^m/x^n = x^{m-n}$	$x^6/x^2 = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2\times3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3y^3$
$(x/y)^n = x^n/y^n$	$(x/y)^2 = x^2 / y^2$
$x^{-n} = 1/x^n$	$x^{-3} = 1/x^3$
And the law about Fractional Exponents:	
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$	$x^{\frac{2}{3}} = \sqrt[3]{x^2} = (\sqrt[3]{x})^2$

Scientific Notation

Scientific Notation is a special way of writing numbers. the number is written in **two parts**:

- Just the **digits** (with the decimal point placed after the first digit), followed by
- **× 10 to a power** that puts the decimal point where it should be
  (i.e. it shows how many places to move the decimal point).



Number	Scientific notation	
700	7 X 100	7 X 10^{2}
4,900,000,000	4.9 X 1000000000	4.9 X 10^{9}
3.1416	3.1416×10^{0}	
3.1416	314.16 X 10^{2}	
6E+5 is the same as 6×10^{5}	6E+5 = 6 × 10 × 10 × 10 × 10 × 10 = 600,000	
0.0000000013	1.3×10^{-9}	
0.000 002 56m	2.56×10^{-6}	
0.000 000 14m	1.4×10^{-7}	
0.0055	$5.5 \times 10^{-3,}$	
0.0055	5.5 × 0.001 = 5.5 × 10-3	
3.2	3.2 × 100	

Engineering Notation

Engineering Notation is like Scientific Notation, except that we only use powers of ten that are multiples of 3 (such as 10^{3}, 10^{-3}, 10^{12} etc).

Examples:

- 2,700 is written $\mathbf{2.7 \times 10^{3}}$
- 27,000 is written $\mathbf{27 \times 10^{3}}$
- 270,000 is written $\mathbf{270 \times 10^{3}}$
- 2,700,000 is written $\mathbf{2.7 \times 10^{6}}$
- 0.00012 is written $\mathbf{120 \times 10^{-6}}$

Types of Equations:

Algebra is a vast subject that studies about the algebraic expressions and equations. An algebraic expression is the combination of constants and variables; where, constants are the fixed quantities and variables are quantities that susceptible to vary. Equations are found everywhere in mathematics. An algebraic equation refers to an algebraic expression with a symbol of equality (=). In other words, an equation is an expression which has an equal to (=) sign between the two algebraic quantities or a set of quantity.
For example - Few algebraic equations are
(1) $4m^{2}n^{2} + 2m = 0$
(2) 7a + 4b + 9c = - 8
(3) $2 \sin A + \cos A = 2\cos^{2} A$

Different types of algebraic equations:

1. Linear equation
2. Quadratic equation
3. Polynomial equation
4. Radical equation
5. Exponential equation
6. Rational equation
7. Trigonometric equation
8. Cubic equation

1) Linear Equations:

A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and a single variable. The graph of linear equation is a straight line if there are two variables.

General form of the linear equation with two variables:

$y = mx + c, m \neq 0$.

Where, **m** is known as slope and **c** at that point on which it cut y axis.

Linear equations with different variables:

a) The equation with one variable: An equation who have only one variable.

Examples:

1. 8a - 8 = 0
2. 9a = 72.

b) The equation with two variables: An equation who have only two types of variable in the equation.

Examples:

1. 7x + 7y = 12
2. 8a - 8d = 74
3. 9a + 6b - 82 = 0.

c) The equation with three variables: An equation who have only three types of variable in the equation.

Examples:

1. 5x + 7y - 6z = 12
2. 13a - 8b + 31c = 74
3. 6p + 14q -7r + 82 = 0.

2) Quadratic Equations:

Quadratic equation is the second degree equation in one variable contains the variable with an exponent of 2.

The general form:

$ax^2 + bx + c = 0$, a $\neq$ 0

Examples of Quadratic Equations:

1. $8x^2 + 7x - 75 = 0$
2. $4y^2 + 14y - 8 = 0$
3. $5a^2 - 5a = 35$

3) Polynomial Equations:

A **polynomial** is an expression consisting of variables (or indeterminates) and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents. An example of a **polynomial** of a single indeterminate x is $x^2 - 4x + 7$.

- **Rational polynomial equations** are of the form $\frac{P(x)}{Q(x)} = 0$, where P(x) and Q(x) are polynomials.
- The **irrational equations** or **radical equations**, are those with the unknown value under the radical sign.

$$\sqrt{2x - 3} - x = -1$$

4) Radical Equations: An equation whose maximum exponent on the variable is $\frac{1}{2}$ and have more than one term or we can say that a radical equation is an equation in which the variable is lying inside a radical symbol usually in a square root.

Examples:

1. $\sqrt{x} + 10 = 26$
2. $\sqrt{x^2 - 5} + x - 1$

5) Exponential Equations: An equation who have variables in the place of exponents. Exponential equation can be solved using the property: a^x = a^y => x = y.

Examples:

1. $a^b = 0$ Here "a" is base and "b" is exponent.
2. $4^x = 0$
3. $8^x = 32$.

6) Rational Equations: A rational equation is one that involves rational expressions.

Example:

$\frac{x}{2}$ = $\frac{x + 2}{4}$.

7) **Trigonometric equations:** an equation involving trigonometric functions of unknown angles

Example: cos B = ½.

$$\begin{array}{ll} \cos\theta = x & \sec\theta = 1/x \ \text{ if } x \neq 0 \\ \sin\theta = y & \csc\theta = 1/y \ \text{ if } y \neq 0 \\ \tan\theta = y/x \ \text{ if } x \neq 0 & \cot\theta = x/y \ \text{ if } y \neq 0 \end{array}$$

A **trigonometric equation** is one in which the unknown to be solved for is an angle (call it θ) and that angle is in the argument of a **trigonometric** function such as sin, cos or tan. A **trigonometric equation** always has an infinite number of solutions, but it is customary to list only those angles between 0° and 360°.

8) Cubic Equations

Cubic equations are equations of the type ax3 + bx2 + cx + d = 0, with a ≠ 0.

9) Other equations:

Differential equations: Differential equation, mathematical statement containing one or more derivatives—that is, terms representing the rates of change of continuously varying quantities.

Integral equations: Integral equation, in mathematics, equation in which the unknown function to be found lies within an integral sign. An example of an integral equation is

$$f(x) = \int_{-\infty}^{\infty} \cos(xt)\, \varphi(t)\, dt,$$

in which *f*(*x*) is known; if *f*(*x*) = *f*(-*x*) for all *x*, one solution is

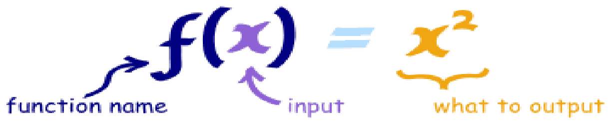
$$\varphi(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos(ux)\, f(u)\, du.$$

Simultaneous equations: System of equations, orsimultaneous equations, in algebra, two or more equations to be solved together (i.e., the solution must satisfy all the equations in the system). A system of linear equations can be represented by a matrix whose elements are the coefficients of the equations. Though simple systems of two equations in two unknowns can be solved by substitution, larger systems are best handled with matrix techniques.

Functions: A function relates an input to an output. It is like a machine that has an input and an output. And the output is related somehow to the input.

In **mathematics**, a **function** is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the **function** that relates each real number x to its square x^2. ... Others are given by a picture, called the graph of the **function**.

f(x)	"**f(x) = ...** " is the classic way of writing a function. And there are other ways, as you will see!



We say *"f of x equals x squared"*. What goes **into** the function is put inside parentheses ( ) after the name of the function:

So ***f(x)*** shows us the function is called "***f***", and "***x***" goes **in**

And we usually see what a function does with the input:

$f(x) = x^2$ shows us that function "***f***" takes "***x***" and squares it.

Example:

- $f(x)=x^2$
- $f(x) = 1 - x + x^2$
- $f(q) = 1 - q + q^2$
- $h(A) = 1 - A + A^2$
- $w(\theta) = 1 - \theta + \theta^2$

The variable (x, q, A, etc) is just there so we know where to put the values: $f(\mathbf{2}) = 1 - \mathbf{2} + \mathbf{2}^2 = 3$

Sometimes a function has no name, and we see something like:

$y = x^2$

But there is still:

- an input (x)
- a relationship (squaring)
- and an output (y)

Function **rules**:

- It must work for **every** possible input value
- And it has only **one relationship** for each input value

Example: $y = x^3$

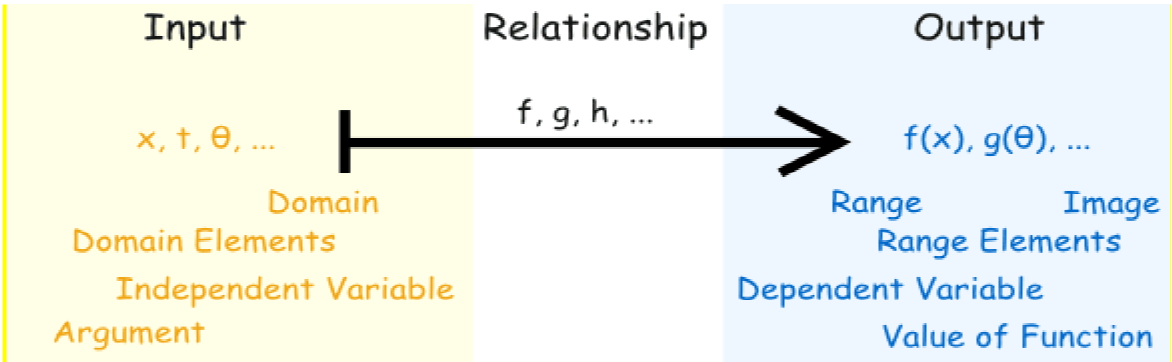
- The input set "X" is all Real Numbers
- The output set "Y" is also all the Real Numbers

few examples:

X: x	Y: x^3
-2	-8
-0.1	-0.001
0	0
1.1	1.331
3	27
and so on...	and so on...

In our examples above

- the set "X" is called the **Domain**,
- the set "Y" is called the **Codomain**, and
- the set of elements that get pointed to in Y (the actual values produced by the function) is called the **Range**.



Conclusion:

- a function **relates** inputs to outputs
- a function takes elements from a set (the **domain**) and relates them to elements in a set (the **codomain**).
- all the outputs (the actual values related to) are together called the **range**
- a function is a **special** type of relation where:
  - **every element** in the domain is included, and
  - any input produces **only one output** (not this **or** that)
- an input and its matching output are together called an **ordered pair**
- so a function can also be seen as a **set of ordered pairs**

Continuous Functions: A function is continuous when its graph is a single unbroken curve
Inverse functions: An inverse function is a function that undoes the action of the another function. A function g is the inverse of a function f if whenever $y=f(x)$ then $x=g(y)$. In other words, applying f and then g is the same thing as doing nothing. We can write this in terms of the composition of f and g as $g(f(x))=x$.

A function f has an inverse function only if for every y in its range there is only one value of x in its domain for which $f(x)=y$. This inverse function is unique and is frequently denoted by f^{-1} and called "f inverse."

Complex functions: Practical applications of functions whose variables are complex numbers are not so easy to illustrate, but they are nevertheless very extensive. They occur, for example, in electrical engineering and aerodynamics. If the complex variable is represented in the form $z = x + iy$, where i is the imaginary unit (the square root of −1) and x and y are real variables (*see* figure), it is possible to split the complex function into real and imaginary parts: $f(z) = P(x, y) + iQ(x, y)$.

Calculus:

Calculus is about changes. The word Calculus comes from **Latin** meaning **"small stone". b**ecause it is like understanding something by looking at small pieces.

Calculus is the branch of mathematics studying the rate of change of quantities (which can be interpreted as slopes of curves) and the length, area, and volume of objects. The calculus is sometimes divided into **differential and integral calculus**, concerned with **derivatives**.

$$\frac{d}{dx} f(x)$$

and integrals

$$\int f(x)\, dx,$$

respectively.

Differential Calculus: Differential calculus is that portion of "the" calculus dealing with derivatives, it cuts something into small pieces to find how it changes.

The derivative of a function represents an *infinitesimal* change in the function with respect to one of its variables.

The "simple" derivative of a function f with respect to a variable x is denoted either $f'(x)$ or

$$\frac{df}{dx},$$

often written in-line as df/dx. When derivatives are taken with respect to time, they are often denoted using Newton's overdot notation for fluxions,

$$\frac{dx}{dt} = \dot{x}.$$

Integral calculus:

An integral is a mathematical object that can be interpreted as an area or a generalization of area. Integrals, together with derivatives, are the fundamental objects of calculus. Other words for integral include anti-derivative and primitive.

Integral Calculus joins (integrates) the small pieces together to find the whole and is known as **integration**.

And Differential Calculus and Integral Calculus are like inverses of each other, just like multiplication and division are inverses.

Application of calculus:

1. Finding the Slope of a Curve
2. Calculating the Area of Any Shape
3. Calculate Complicated X-intercepts
4. Visualizing Graphs
5. Finding the Average of a Function
6. Calculating Optimal Values

Introduction to differentiation and differential calculus:

Differential Calculus: Differential calculus is that portion of "the" calculus dealing with derivatives, it cuts something into small pieces to find how it changes.

The derivative of a function represents an *infinitesimal* change in the function with respect to one of its variables.

The "simple" derivative of a function f with respect to a variable x is denoted either $f'(x)$ or

$$\frac{df}{dx},$$

often written in-line as df/dx. When derivatives are taken with respect to time, they are often denoted using Newton's overdot notation for fluxions,

$$\frac{dx}{dt} = \dot{x}.$$

Notations: derivative is all about change, the definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change). It can be written in various methods, some of them are as under:

1. $\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}} = \frac{\Delta y}{\Delta x}$
2. $\frac{d}{dx}$,
3. f'(x)

Higher order of derivatives: If *f*(*x*) is a differentiable function, then its derivative *f*′(*x*) is also a function, so may have a derivative (finite or not). This function is called second derivative of *f*(*x*) because it is derivative of derivative and denoted by *f*′′ . So, *f*′′=(*f*′)′.

Following notations are used for second derivative: $\frac{d^2y}{dx^2}, \frac{d^2}{dx^2}(f(x)), y'', f''(x), D^2y, D^2f(x)$.

Example 3. Find fourth derivative of $y = \frac{1}{2}x^4 - \frac{1}{6}x^3 + 2x^2 + \frac{4}{3}x - \frac{1}{2}$.

We have that $y' = 2x^3 - 2x^2 + 4x + \frac{4}{3}$.

$$y'' = (y')' = \left(2x^3 - 2x^2 + 4x + \frac{4}{3}\right)' = 6x^2 - 4x + 4.$$

$$y''' = (y'')' = (6x^2 - 4x + 4)' = 12x - 4.$$

$$y^{(4)} = (y''')' = (12x - 4)' = 12.$$

Note, that derivatives of order higher than 4 will be zero in this case.

Derivative rule:

Rules	Function	Derivative
Multiplication by constant	cf	cf'
Power Rule	x^n	nx^{n-1}
Sum Rule	$f + g$	$f' + g'$
Difference Rule	$f - g$	$f' - g'$
Product Rule	fg	$f g' + f' g$
Quotient Rule	f/g	$(f' g - g' f)/g^2$
Reciprocal Rule	$1/f$	$-f'/f^2$
Chain Rule (as "Composition of Functions")	$f \circ g$	$(f' \circ g) \times g'$
Chain Rule (using ′)	$f(g(x))$	$f'(g(x))g'(x)$
Chain Rule (using $\frac{d}{dx}$)	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	

Common Functions	Function	Derivative
Constant	c	0
Line	x	1
	ax	a
Square	x^2	$2x$
Square Root	$\sqrt{x}$	$(½)x^{-½}$
Exponential	e^x	e^x
	a^x	$\ln(a)\, a^x$
Logarithms	$\ln(x)$	$1/x$
	$\log_a(x)$	$1 / (x \ln(a))$
Trigonometry (x is in radians)	$\sin(x)$	$\cos(x)$
	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\sec^2(x)$
Inverse Trigonometry	$\sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
	$\cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$
	$\tan^{-1}(x)$	$1/(1+x^2)$

Example problems: first order derivative

What is $\frac{d}{dx}x^3$? The question is asking "what is the derivative of x^3?" We can use the Power Rule, where n=3: $\frac{d}{dx}x^n = nx^{n-1}$ $\frac{d}{dx}x^3 = 3x^{3-1} = \mathbf{3x^2}$	**What is $\frac{d}{dx}(1/x)$?** 1/x is also $\mathbf{x^{-1}}$ We can use the Power Rule, where $n = -1$: $\frac{d}{dx}x^n = nx^{n-1}$ $\frac{d}{dx}x^{-1} = -1x^{-1-1} = \mathbf{-x^{-2}}$
What is $\frac{d}{dx}5x^3$? the derivative of cf = cf' the derivative of 5f = 5f' We know (from the Power Rule): $\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$ So: $\frac{d}{dx}5x^3 = 5\frac{d}{dx}x^3 = 5 \times 3x^2 = \mathbf{15x^2}$	**What is the derivative of x^2+x^3 ?** The Sum Rule says: the derivative of f + g = f' + g' So we can work out each derivative separately and then add them. Using the Power Rule: • $\frac{d}{dx}x^2 = 2x$ • $\frac{d}{dx}x^3 = 3x^2$ And so: the derivative of $x^2 + x^3 = \mathbf{2x + 3x^2}$
What is $\frac{d}{dv}(v^3-v^4)$? The Difference Rule says the derivative of f − g = f' − g' So we can work out each derivative separately and then subtract them. Using the Power Rule: • $\frac{d}{dv}v^3 = 3v^2$ • $\frac{d}{dv}v^4 = 4v^3$ And so: the derivative of $v^3 - v^4 = \mathbf{3v^2 - 4v^3}$	**What is $\frac{d}{dz}(5z^2 + z^3 - 7z^4)$?** Using the Power Rule: • $\frac{d}{dz}z^2 = 2z$ • $\frac{d}{dz}z^3 = 3z^2$ • $\frac{d}{dz}z^4 = 4z^3$ And so: $\frac{d}{dz}(5z^2 + z^3 - 7z^4) = 5 \times 2z + 3z^2 - 7 \times 4z^3 =$ $\mathbf{10z + 3z^2 - 28z^3}$

What is the derivative of cos(x)sin(x) ?

The Product Rule says:
the derivative of fg = f g’ + f’ g
In our case:

- f = cos
- g = sin

We know (from the table above):

- $\frac{d}{dx}$ cos(x) = −sin(x)
- $\frac{d}{dx}$ sin(x) = cos(x)

So:the derivative of cos(x)sin(x) = cos(x)cos(x) − sin(x)sin(x)

= **$\cos^2(x) - \sin^2(x)$**

What is $\frac{d}{dx}$ (1/x) ?

The Reciprocal Rule says:

the derivative of 1/f = −f’/f^2

With f(x)= x, we know that f’(x) = 1

So:

the derivative of 1/x = −1/x^2

Which is the same result we got above using the Power Rule.

What is *d* dx sin(x^2) ?

sin(x^2) is made up of **sin()** and **x^2**:

- f(g) = sin(g)
- g(x) = x^2

The Chain Rule says:
the derivative of f(g(x)) = f'(g(x))g'(x)
The individual derivatives are:

- f'(g) = cos(g)
- g'(x) = 2x

So:
d **dx** sin(x^2) = cos(g(x)) (2x)
= 2x cos(x^2)

What is *d* dx sin(x^2) ?

dy **dx** = *dy* **du** *du* **dx**

Have u = x^2, so y = sin(u):

d **dx** sin(x^2) = *d* **du** sin(u) *d* **dx** x^2

Differentiate each:

d **dx** sin(x^2) = cos(u) (2x)

Substitue back u = x^2 and simplify:

d **dx** sin(x^2) = 2x cos(x^2)

What is $\frac{d}{dx}$ (1/cos(x)) ?

1/cos(x) is made up of **1/g** and **cos()**:

- f(g) = 1/g
- g(x) = cos(x)

The Chain Rule says:

the derivative of f(g(x)) = f’(g(x))g’(x)

The individual derivatives are:

- f'(g) = −1/(g^2)

What is $\frac{d}{dx}$ $(5x-2)^3$?

The Chain Rule says:

the derivative of f(g(x)) = f’(g(x))g’(x)

$(5x-2)^3$ is made up of **g^3** and **5x-2**:

- f(g) = g^3
- g(x) = 5x−2

The individual derivatives are:

- f'(g) = $3g^2$ (by the Power Rule)

• $g'(x) = -\sin(x)$ So: **$(1/\cos(x))' = -1/(g(x))^2 \times -\sin(x)$** **$= \sin(x)/\cos^2(x)$** Note: **$\sin(x)/\cos^2(x)$** is also **$\tan(x)/\cos(x)$**, or many other forms.	• $g'(x) = 5$ So: **$\frac{d}{dx}(5x-2)^3 = 3g(x)^2 \times 5 = 15(5x-2)^2$**

Application of derivatives: Derivatives are used to find,

1. **Rates of Change**	2. **Critical Points**
3. **Minimum and Maximum Values**	4. **Finding Absolute Extrema**
5. **The Shape of a Graph**	6. **The Mean Value Theorem**
7. **Optimization Problems**	8. **More Optimization Problems**
9. **L'Hospital's Rule and Indeterminate Forms**	10. **Linear Approximations**
11. **Differentials**	12. **Business Applications**

Differential equations:

Differential equation, mathematical statement containing one or more derivatives—that is, terms representing the rates of change of continuously varying quantities. Differential equations are very common in science and engineering, as well as in many other fields of quantitative study, because what can be directly observed and measured for systems undergoing changes are their rates of change. The solution of a differential equation is, in general, an equation expressing the functional dependence of one variable upon one or more others; it ordinarily contains constant terms that are not present in the original differential equation. Another way of saying this is that the solution of a differential equation produces a function that can be used to predict the behaviour of the original system, at least within certain constraints.

Differential equations are classified into several broad categories, and these are in turn further divided into many subcategories. The most important categories are

ordinary differential equations When the function involved in the equation depends on only a single variable, its derivatives are ordinary derivatives and the differential equation is classed as an ordinary differential equation.

partial differential equations. On the other hand, if the function depends on several independent variables, so that its derivatives are partial derivatives, the differential equation is classed as a partial differential equation. The following are examples of ordinary differential equations:

$$\frac{dy}{dt} = -ky,$$

$$m\frac{d^2y}{dt^2} = -k^2y,$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0.$$

In these, *y* stands for the function, and either *t* or *x* is the independent variable. The symbols *k* and *m* are used here to stand for specific constants.

Whichever the type may be, a differential equation is said to be of the *n*th order if it involves a derivative of the *n*th order but no derivative of an order higher than this. The equation

$$\frac{\partial u}{\partial t} = k^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

is an example of a partial differential equation of the second order. The theories of ordinary and partial differential equations are markedly different, and for this reason the two categories are treated separately.

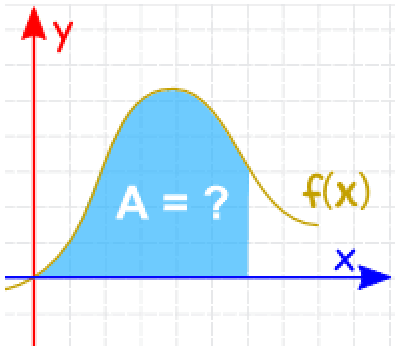
Instead of a single differential equation, the object of study may be a simultaneous system of such equations. The formulation of the laws of dynamics frequently leads to such systems. In many cases, a single differential equation of the *n*th order is advantageously replaceable by a system of *n* simultaneous equations, each of which is of the first order, so that techniques from linear algebra can be applied.

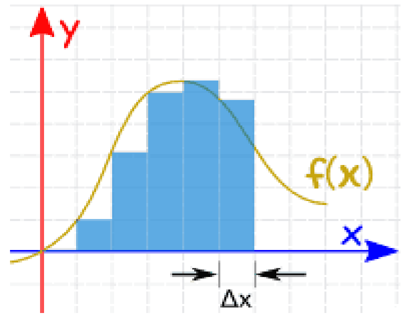
An ordinary differential equation in which, for example, the function and the independent variable are denoted by *y* and *x* is in effect an implicit summary of the essential characteristics of *y* as a function of *x*. These characteristics would presumably be more accessible to analysis if an explicit formula for *y* could be produced. Such a formula, or at least an equation in *x* and *y* (involving no derivatives) that is deducible from the differential equation, is called a solution of the differential equation. The process of deducing a solution from the equation by the applications of algebra and calculus is called solving or integrating the equation. It should be noted, however, that the differential equations that can be explicitly solved form but a small minority. Thus, most functions must be studied by indirect methods. Even its existence must be proved when there is no possibility of producing it for inspection. In practice, methods from numerical analysis, involving computers, are employed to obtain useful approximate solutions.

Introduction to Integration

Integration is a way of adding slices to find the whole. The process of finding a function, given its derivative is called anti-differentiation or integration. If f`(x)=f(x), we say f(x) is an anti-derivative of f``(x).

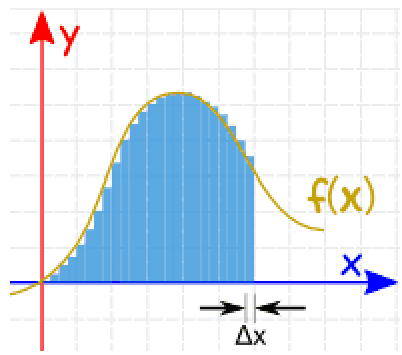
Integration can be used to find areas, volumes, central points and many useful things. But it is easiest to start with finding the **area under the curve of a function** as follows:



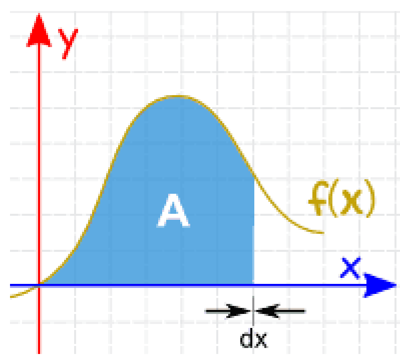


Slices or integral: An integral is a mathematical object that can be interpreted as an area or a generalization of area. Integrals, together with derivatives, are the fundamental objects of calculus. Other words for integral include anti-derivative and primitive.

We could calculate the function at a few points and **add up slices of width Δx** like this (but the answer won't be very accurate):



We can make **Δx** a lot smaller and **add up many small slices** (answer is getting better):



And as the slices **approach zero in width**, the answer approaches the **true answer**.

We now write **dx** to mean the **Δx** slices are approaching zero in width.

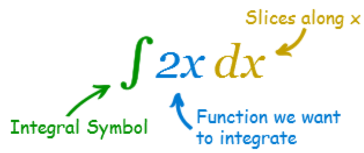
Note: finding an Integral is the reverse **of finding a Derivative.**

Example: integral of 2x

integral of 2x is x^2 , as the derivative of x^2 is 2x

Notation:

The symbol for "Integral" is a stylish "S" (for "Sum", the idea of summing slices):



After the Integral Symbol we put the function we want to find the integral of (called the Integrand), and then finish with **dx** to mean the slices go in the x direction (and approach zero in width).

And here is how we write the answer:

$$\int 2x \, dx = x^2 + C$$

Here C is the "Constant of Integration". It is there because of **all the functions whose derivative is 2x**: for example, The derivative of $\mathbf{x^2+4}$ is **2x**, and the derivative of $\mathbf{x^2+99}$ is also **2x**, and so on, Because the derivative of a constant is zero.

So when we **reverse** the operation (to find the integral) we only know **2x**, but there could have been a constant of any value. So we wrap up the idea by just writing + C at the end.

Rules of integration:

Examples			
Sl no	**Common Functions**	**Function**	**Integral**
1	Constant	$\int a\,dx$	$ax + C$
2	Variable	$\int x\,dx$	$x^2/2 + C$
3	Square	$\int x^2\,dx$	$x^3/3 + C$
4	Reciprocal	$\int (1/x)\,dx$	$\ln\|x\| + C$
5	Exponential	$\int e^x\,dx$	$e^x + C$
		$\int a^x\,dx$	$a^x/\ln(a) + C$
		$\int \ln(x)\,dx$	$x \ln(x) - x + C$
6	Trigonometry (x in radians)	$\int \cos(x)\,dx$	$\sin(x) + C$
		$\int \sin(x)\,dx$	$-\cos(x) + C$
		$\int \sec^2(x)\,dx$	$\tan(x) + C$
	Rules	**Function**	**Integral**
1	Multiplication by constant	$\int cf(x)\,dx$	$c\int f(x)\,dx$
2	Power Rule ($n \neq -1$)	$\int x^n\,dx$	$x^{n+1}/(n+1) + C$
3	Sum Rule	$\int (f + g)\,dx$	$\int f\,dx + \int g\,dx$

4	Difference Rule	∫(f - g) dx	∫f dx - ∫g dx
5	Integration by Parts		
6	Substitution Rule		

Integration by parts:

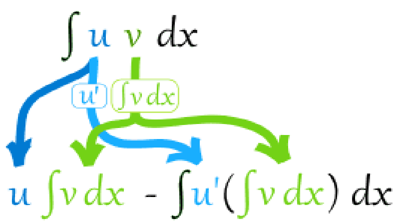
Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

You will see plenty of examples soon, but first let us see the rule:

$\int u\, v\, dx = u\int v\, dx - \int u' \left(\int v\, dx\right) dx$

- **u** is the function u(x)
- **v** is the function v(x)

As a diagram:



Integration by Substitution

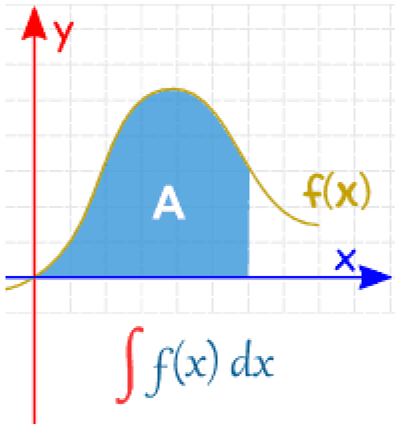
"Integration by Substitution" (also called "u-substitution") is a method to find an integral, but only when it can be set up in a special way.

The first and most vital step is to be able to write our integral in this form:

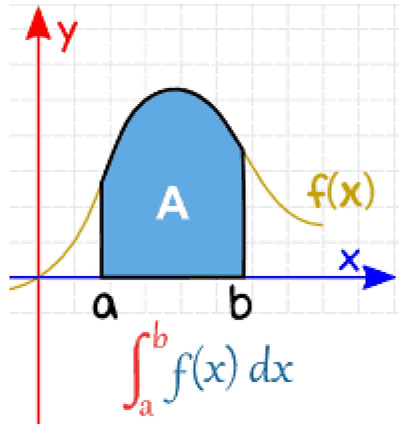
Note that we have **g(x)** and its derivative **g'(x)**

Definite and Indefinite Integrals

Definite Integral has actual values to calculate between (they are put at the bottom and top of the "S"):



Indefinite Integral



Definite Integral

A **Definite Integral** has start and end values: in other words there is an **interval** (a to b).

Analytical integrations or symbolic are exact and obtained by methods of symbolic manipulation, derived using analysis.

Numerical analysis usually indicates an approximate solution obtained by methods of numerical analysis.

Numerical integration: In numerical analysis, numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral and by extension, the term is also sometimes used to describe the numerical solution of differential equations.

Numerical integration is the approximate computation of an integral using numerical techniques, it is also called **quadrature.**

Methods: the choice of methods depends in part on the results required and in part on the function or data to be integrated

- Manual method
- Trapezium rule
- Midpoint rule
- Simpson's rule
- Quadratic triangulation
- Romberg integration
- Guass quadrature.

Applications of integration: integration is used to find

1. Area under a curve
2. Area between 2 curves
3. Volume of solid of revolution
4. Centroid of an area
5. Moments of inertia
6. Work by a variable force
7. Displacement, velocity etc

Simultaneous Equations: are also known as **system of equations**.

Definition:

1. A set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the number of variables being equal to or less than the number of equations in the set.
2. A set of **simultaneous equations**, also known as a **system of equations** or an **equation system**, is a finite set of equations for which common solutions are sought.

They are called simultaneous because they must be solved at the same time. An equation system is usually classified in the same manner as single equations, namely as a:

- System of linear equations
- System of bilinear equations
- System of polynomial equations
- System of ordinary differential equations
- System of partial differential equations
- System of difference equations

System of linear equations or simultaneous linear equations:

A system of linear equations is a collection of 2 or more linear equations involving the same set of variables.

Example

3x+2y-z=1

2x-2y+4z=-2

-x+ $\frac{1}{2}$ y-z=0, here there are 3 variables x,y,z. A solution to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied and the above system solution is x=1, y=-2, z=-2.

Representation of linear equations:

1. **General form :**

A general system of *m* linear equations with *n* unknowns can be written as

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & = b_m. \end{array}$$

Here X_1, X_2, ….., X_n are the unknowns, a_{11}, a_{12}, ……., a_{mn} are the coefficients of the system, and b_1, b_2….b_m are the constant terms.

Often the coefficients and unknowns are real or complex numbers, but integers and rational numbers are also seen, as are polynomials and elements of an abstract algebraic structure.

2. **Vector form**

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This allows all the language and theory of *vector spaces* (or more generally, *modules*) to be brought to bear. For example, the collection of all possible linear combinations of the vectors on the left-hand side is called their *span*, and the equations have a solution just when the right-hand vector is within that span. If every vector within that span has exactly one expression as a linear combination of the given left-hand vectors, then any solution is unique. In any event, the span has a *basis* of linearly independent vectors that do guarantee exactly one expression; and the number of vectors in that basis (its *dimension*) cannot be larger than *m* or *n*, but it can be smaller. This is important because if we have *m* independent vectors a solution is guaranteed regardless of the right-hand side, and otherwise not guaranteed.

3. **Matrix form:** The vector equation is equivalent to a matrix equation of the form

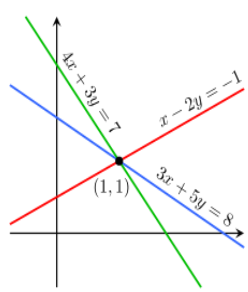
$$A\mathbf{x} = \mathbf{b}$$

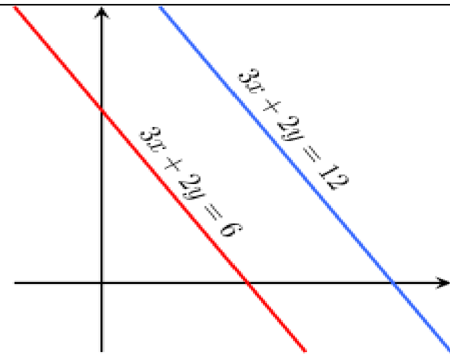
where A is an $m \times n$ matrix, $\mathbf{x}$ is a column vector with n entries, and $\mathbf{b}$ is a column vector with m entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The number of vectors in a basis for the span is now expressed as the *rank* of the matrix.

Properties:

1. Independence: the equations of a linear system are independent if none of the equation can be derived algebraically from the others. Example: $x - 2y = -1$ $3x + 5y = 8$ $4x + 3y = 7$	 The equations $x - 2y = -1$, $3x + 5y = 8$, and $4x + 3y = 7$ are linearly dependent.

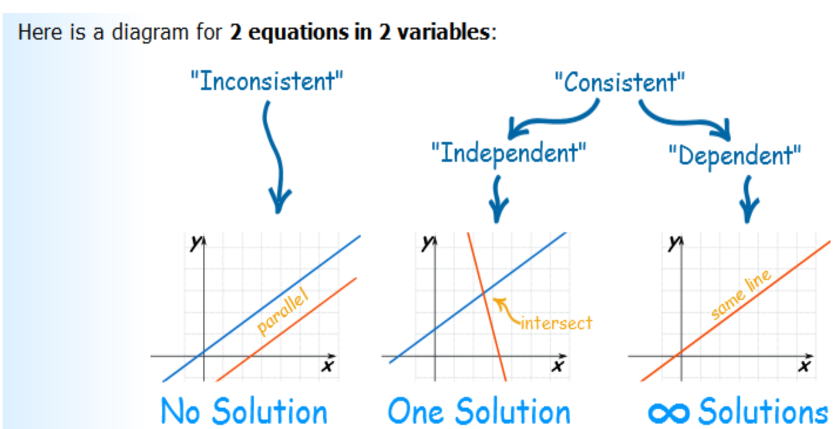
2. Consistency A linear system is **inconsistent** if it has no solution, and otherwise it is said to be **consistent**. When the system is inconsistent, it is possible to derive a contradiction from the equations, that may always be rewritten as the statement 0 = 1. For example, the equations $3x + 2y = 6$ and $3x + 2y = 12$ are inconsistent.	 The equations $3x + 2y = 6$ and $3x + 2y = 12$ are inconsistent
3. Equivalence Two linear systems using the same set of variables are equivalent if each of the equations in the second system can be derived algebraically from the equations in the first system, and vice versa.	

Solving a linear equation: When the number of equations is the **same** as the number of variables, there is **likely** to be a solution. Not guaranteed, but likely. There are only three possible cases:

- **No** solution
- **One** solution
- **Infinitely many** solutions

When there is **no solution** the equations are called **"inconsistent"**. **One** or **infinitely many solutions** are called **"consistent".**

Here is a diagram for **2 equations in 2 variables**:



"Independent" means that each equation gives new information. Otherwise they are **"Dependent"**. Also called "Linear Independence" and "Linear Dependence".

Methods of solving systems of linear equations:

The different methods of solving simultaneous linear equations can be broadly grouped as,

1. Graphical solution
2. Algebraic solution
   - Describing the solution
   - Evaluation of determinants:
   - **Direct method**
     - Example: Solving by elimination – guass elimination, guass Jordan elimination.
   - **Iterative method**
     - Example : Solving by substitution
   - Matrix solution:
   - Using computers
   - Pivoting
   - LU Decomposition etc.
3. Manual

1. Graphical solution:

The **graphical solution** of linear simultaneous equations is the point of intersection found by drawing the two linear equations on the same axes.

Example 1

Solve the following simultaneous equations graphically.

$x + y = 8$

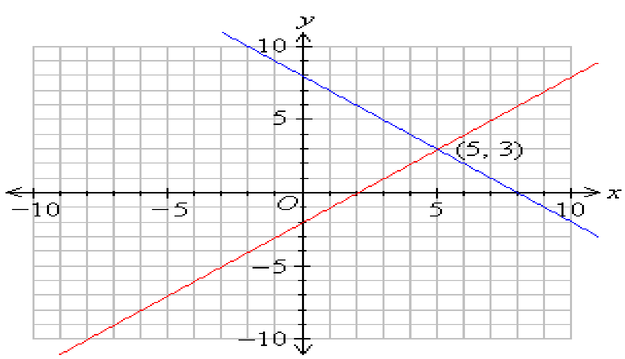
$x - y = 2$

Solution:

The graphical solution of the simultaneous equations

$x + y = 8$ and $x - y = 2$ is given by the point of intersection of the linear equations.

Consider $x + y = 8$. x-intercept: When $y = 0$, $x = 8$ y-intercept: When $x = 0$, $y = 8$	Consider $x – y = 2$. x-intercept: When $y = 0$, $x = 2$ y-intercept: When $x = 0$, $-y = 2$ $\therefore y = -2$



The diagram shows that the lines intersect at the point (5, 3). So, the solution of the simultaneous equations is $x = 5$ and $y = 3$ or (5, 3).

Note: Often the answer obtained with the graphical method is not exact.

2. Describing the solution

When the solution set is finite, it is reduced to a single element. In this case, the unique solution is described by a sequence of equations whose left-hand sides are the names of the unknowns and right-hand sides are the corresponding values, for example $(x = 3,\ y = -2,\ z = 6)$ When an order on the unknowns has been fixed, for example the alphabetical order the solution may be described as a vector of values, like $(3,\ -2,\ 6)$. It can be difficult to describe a set with infinite solutions. Typically, some of the variables are designated as **free** (or **independent**, or as **parameters**), meaning that they are allowed to take any value, while the remaining variables are **dependent** on the values of the free variables.

3. **Evaluation of determinants:** This is satisfactory for small number of simultaneous equations. Evaluation of determinants also called carmer's rule.

4. Direct method: the direct methods solve the problem by a finite seqyence of operations. In absence of rounding errors, direct method would deliver an exact solution.

Example: **Elimination of variables – guassian elimination (row reduction), guassian Jordan etc**

The direct method is based on elimination of variables to transform the set of equations to a triangular form.

Elimination: "Eliminate" means to **remove**: this method works by removing variables until there is just one left. The simplest method for solving a system of linear equations is to repeatedly eliminate variables. This method can be described as follows:

1. In the first equation, solve for one of the variables in terms of the others.
2. Substitute this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.
3. Repeat until the system is reduced to a single linear equation.
4. Solve this equation and then back-substitute until the entire solution is found.

Guass elimination: This method for solving a pair of simultaneous linear equations reduces one equation to one that has only a single variable. Once this has been done, the solution is the same as that for when one line was vertical or parallel. This method is known as the ***Gaussian elimination method***. This method is sometimes referred as elimination by adding.

Steps: Gaussian elimination is summarized by the following three steps:

1. Write the system of equations in matrix form. Form the augmented matrix. You omit the symbols for the variables, the equal signs, and just write the coefficients and the unknowns in a matrix. You should consider the matrix as shorthand for the original set of equations.
2. Perform elementary row operations to get zeros below the diagonal.
3. An elementary row operation is one of the following:

- multiply each element of the row by a non-zero constant
- Switch two rows
- add (or subtract) a non-zero constant times a row to another row

4. Inspect the resulting matrix and re-interpret it as a system of equations.
If you get 0 = a non-zero quantity then there is no solution.

- If you get less equations than unknowns after discarding equations of the form 0=0 and if there isa solution then there is an infinite number of solutions
- If you get as many equations as unknowns after discarding equations of the form 0=0 and if there is a solution then there is exactly one solution

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Examples:

$$\begin{aligned} x+y+z &= 6 \\ 2x-y+z &= 3 \\ x+z &= 4 \end{aligned}$$

First form the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

—

Next add -2 times the first row to the second row and then add -1 times the first row to the third row:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & 0 & -2 \end{pmatrix}$$

Next multiply the second row by -1 and the third row by -1, just to get rid of the minus signs. Then switch the second and third rows:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 9 \end{pmatrix}$$

Now add -3 times the second row to the third row, so we have all zeros below the diagonal:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Now re-interpret the augmented matrix as a system of equations, starting at the bottom and working backwards (this is called back substitution).

The bottom equation is $0x + 0y + z = 3$ so $z = 3$.

The next to the bottom equation is $0x + y + 0z = 2$ so $y = 2$.

The next equation (the top one) is $x + y + z = 6$. Substitute the values $z = 3$ and $y = 2$ into the equation and get $x = 1$.

Guass Jordan Elimination: The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.
1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form described below, which is called the reduced row echelon form (RREF).
(a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
(b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
(c) Each column that contains a leading 1 has zeros in all other entries.
(d) The leading 1 in any row is to the left of any leading 1's in the rows below it.
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.
Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

5. Iterative method: it is a successive approximation procedure.

Types:

1. Stationary iterative methods
   a) Jacobi method
   b) Gauss seidel method
   c) Successive over relaxation method

2. Krylov subspace methods
   a) conjurate gradient method (CG)
   b) generalized minimal residual method (GMBES)
   c) Bi_Conjurate gradient method.

Gauss seidel method: Given a general set of equations and unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$
$$\vdots \qquad \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with on the left hand side, the second equation is rewritten with on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

.

.

.

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\in_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

where x_i^{new} is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.
When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example:

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations given in Example 1.

The first computation is identical to that given in Example 1. That is, using $(x_1, x_2, x_3) = (0, 0, 0)$ as the initial approximation, you obtain the following new value for x_1.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$

Now that you have a new value for x_1, however, use it to compute a new value for x_2. That is,

$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(0) \approx 0.156.$$

Similarly, use $x_1 = -0.200$ and $x_2 = 0.156$ to compute a new value for x_3. That is,

$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.156) \approx -0.508.$$

So the first approximation is $x_1 = -0.200$, $x_2 = 0.156$, and $x_3 = -0.508$. Continued iterations produce the sequence of approximations shown in Table 10.2.

TABLE 10.2

n	0	1	2	3	4	5
x_1	0.000	−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

6. Other methods:

- Matrix solution:
- Using computers
- Manual
- Pivoting
- LU Decomposition etc.

Pivoting:

Pivot element is the element of a matrix or an array, which is selected first by an algorithm (ex: Gaussian elimination, simplex algorithm etc), to do certain calculations. In the case of matrix algorithms, a pivot entry is usually required to be at least distinct from zero, and often distant from it, in this case finding this element is called pivoting.

Pivoting may be followed by an interchange of rows or columns to bring the pivot to a fixe position and allow the algorithm to proceed successfully and to reduce the round off error.

Pivoting may be thought of as swapping or sorting rows or column in a matrix and thus can be represented as multiplication by permutation matrices.

Over all pivoting adds more operations to the computational cost of an algorithm.

Refer Gaussian Elimination with pivoting problem example:

LU Decomposition method (LU Factorization):

A square matrix is said to be **lower triangular** matrix if the entries above the principle diagonal is zero.

A square matrix is said to be **upper triangular** matrix if the entries below the principle diagonal is zero.

A lower triangular matrix is said **to unit lower triangular matrix** if the principle diagonal of lower triangle matrix is equal to 1.

In numerical analysis and linear algebra, LU decomposition (where 'LU' stands for '**lower upper'**, and also called LU factorization) factors a matrix as the product of a lower triangular matrix and an upper triangular matrix. The product sometimes includes a permutation matrix as well. The LU decomposition can be viewed as the matrix form of Gaussian elimination.

Definition: Let *A* be a square matrix. An **LU factorization** refers to the factorization of A, with proper row and/or column orderings or permutations, into two factors, a lower triangular matrix *L* and an upper triangular matrix *U*,

$$A=LU$$

In the lower triangular matrix all elements above the diagonal are zero, in the upper triangular matrix, all the elements below the diagonal are zero. For example, for a 3-by-3 matrix A, its LU decomposition looks as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Without a proper ordering or permutations in the matrix, the factorization may fail to materialize. For example, it is easy to verify (by expanding the matrix multiplication) that $a_{11} = l_{11}u_{11}$. If $a_{11} = 0$, then at least one of l_{11} and u_{11} has to be zero, which implies either *L* or *U* is singular. This is impossible if *A* is nonsingular. This is a procedural problem. It can be removed by simply reordering the rows of *A* so that the first element of the permuted matrix is nonzero. The same problem in subsequent factorization steps can be removed the same way; see the basic procedure below.

Steps:

Consider system of equations:

$$\left(\begin{array}{l} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array} \right)$$

This equation may be written as AX = b

$$A = \left(\begin{array}{l} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array} \right) \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} b1 \\ b2 \\ b3 \end{pmatrix}$$

Let A be a non singular matrix and let A=LU where L=unit lower triangle matrix and U=upper triangle matrix i.e.,

$$L = \begin{matrix} 1 & 0 & 0 \\ L21 & 1 & 0 \\ L31 & L32 & 1 \end{matrix} \qquad U = \begin{matrix} U11 & U12 & U13 \\ 0 & U22 & U23 \\ 0 & 0 & U33 \end{matrix}$$

We have system or equation AX=B.

W.K.T A=LU therefore LUX = B -------> 1

Put UX = Y where

$$Y = \begin{matrix} Y1 \\ Y2 \\ y3 \end{matrix}$$

are some unknowns

Therefore equation 1 becomes LY = B and we have UX = Y. first solve LY = B for Y using forward substitution method and then solve UX =Y for X using backward substitution.

<u>Refer problems:</u>

Difference between direct and iterative methods:

"Direct" techniques use a "formula", whereas "indirect" techniques iterate until convergence. The criteria considered ***are time to converge, number of iterations, memory requirements and accuracy.*** In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

In direct method, one has to do a certain amount of work, till the end to be close to the exact solution. And if the working is incomplete, then, the solution is approximate.

In iterative method you can decide how much work you want to invest depending on how accurate you need your solution.

The second big difference is that, for a direct method, you generally need to have the entire matrix stored in memory. Not so in iterative methods: here, you typically only need a way to compute the application of the matrix to a vector, i.e., the matrix-vector product, So if you have a very large matrix, but you can relatively quickly compute its application to a vector (for instance, because the matrix is very sparse or has some other kind of structure), you can use this to your advantage.

Ill conditioned equations:

For any system of linear equations, the question of how many errors are there in a solution obtained by a numerical method is not readily answered.

1) Errors in the co-efficient and constants: in many practical cases, the co-efficient of the variables and also the constants on the RHS of the equation are obtained from observations of experiments or from other numerical calculations. They will have errors and hence the solution will too contain error.
2) Round off errors and number of operations:- numerical methods for solving systems of linear equations involve large number of arithmetic operations. Since round off errors are

propagated at each step of an algorithm, the growth of round off errors can be such that, when n is large, a solution differs greatly from the true one.

ILL conditioning:

Certain systems of linear equations are such that their solution are very sensitive to small changes and therefore to errors, in their coefficients and constants. Such system are said to be ill conditioned.

If a system is ill conditioned, a solution obtained by a numerical method may be differ greatly from exact solution, even through great case is taken to keep round off and other errors very small.

Example: consider $2x + y = 4$

$$2x + 1.01y = 4.02$$

The exact solution is x=1, y=2. Suppose the coefficients of 2nd equation is changed by 1% and the constant of the 1st equation by 5%, this yields $2x + y = 3.8$ and $2.02x + y = 4.02$

Note: partial pivoting helps in reducing of errors.

Some definitions:

Diagonal dominancy: In mathematics, a square matrix is said to be **diagonally dominant** if for every row of the matrix, the magnitude of the **diagonal** entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-**diagonal**) entries in that row.

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Tabulation: The process of placing classified data into tabular form is known as **tabulation**. A table is a symmetric arrangement of **statistical** data in rows and columns. Rows are horizontal arrangements whereas columns are vertical arrangements.