Unit 2: chapter 2:

Solving simultaneous linear equations: Introduction, Gauss Elimination method, pivoting, ill conditioned equations, Gauss Jordon method, LU Decomposition method and Gauss-Seidel iterative method. Comparison of direct and iterative methods.

Simultaneous Equations: are also known as **system of equations**.

Definition:

1. A set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the number of variables being equal to or less than the number of equations in the set.
2. A set of **simultaneous equations**, also known as a **system of equations** or an **equation system**, is a finite set of equations for which common solutions are sought.

They are called simultaneous because they must be solved at the same time. An equation system is usually classified in the same manner as single equations, namely as a:

- System of linear equations
- System of bilinear equations
- System of polynomial equations
- System of ordinary differential equations
- System of partial differential equations
- System of difference equations

System of linear equations or simultaneous linear equations:

A system of linear equations is a collection of 2 or more linear equations involving the same set of variables.

Example

3x+2y-z=1
2x-2y+4z=-2
-x+ $\frac{1}{2}$ y-z=0, here there are 3 variables x,y,z. A solution to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied and the above system solution is x=1, y=-2, z=-2.

Representation of linear equations:

1. **General form :**

A general system of *m* linear equations with *n* unknowns can be written as

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 &+\cdots+ a_{1n}x_n &= b_1 \\ a_{21}x_1 &+ a_{22}x_2 &+\cdots+ a_{2n}x_n &= b_2 \\ \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 &+\cdots+ a_{mn}x_n &= b_m. \end{aligned}$$

Here X_1, X_2, ….., X_n are the unknowns, a_{11}, a_{12}, ……., a_{mn} are the coefficients of the system, and b_1, b_2….b_m are the constant terms.

Often the coefficients and unknowns are real or complex numbers, but integers and rational numbers are also seen, as are polynomials and elements of an abstract algebraic structure.

2. **Vector form**

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination.

$$x_1\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}+x_2\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}+\cdots+x_n\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}$$

This allows all the language and theory of *vector spaces* (or more generally, *modules*) to be brought to bear. For example, the collection of all possible linear combinations of the vectors on the left-hand side is called their *span*, and the equations have a solution just when the right-hand vector is within that span. If every vector within that span has exactly one expression as a linear combination of the given left-hand vectors, then any solution is unique. In any event, the span has a *basis* of linearly independent vectors that do guarantee exactly one expression; and the number of vectors in that basis (its *dimension*) cannot be larger than *m* or *n*, but it can be smaller. This is important because if we have *m* independent vectors a solution is guaranteed regardless of the right-hand side, and otherwise not guaranteed.

3. **Matrix form:** The vector equation is equivalent to a matrix equation of the form

$$A\mathbf{x}=\mathbf{b}$$

where *A* is an *m*×*n* matrix, **x** is a column vector with *n* entries, and **b** is a column vector with *m* entries

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix},\quad \mathbf{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}$$

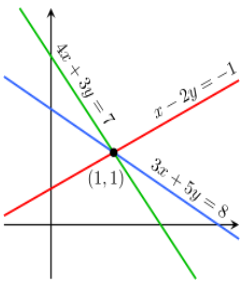
The number of vectors in a basis for the span is now expressed as the *rank* of the matrix.

Properties:

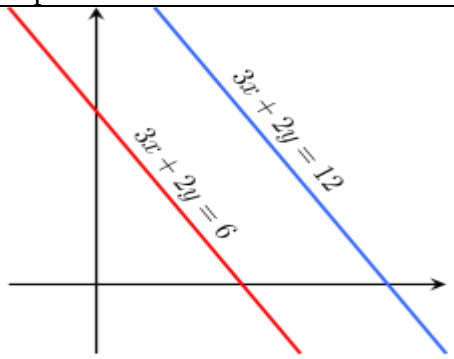
1. Independence: the equations of a linear system are independent if none of the equation can be derived algebraically from the others.

Example:

$$\begin{aligned}x-2y&=-1\\3x+5y&=8\\4x+3y&=7\end{aligned}$$



The equations $x - 2y = -1$, $3x + 5y = 8$, and $4x + 3y = 7$ are linearly

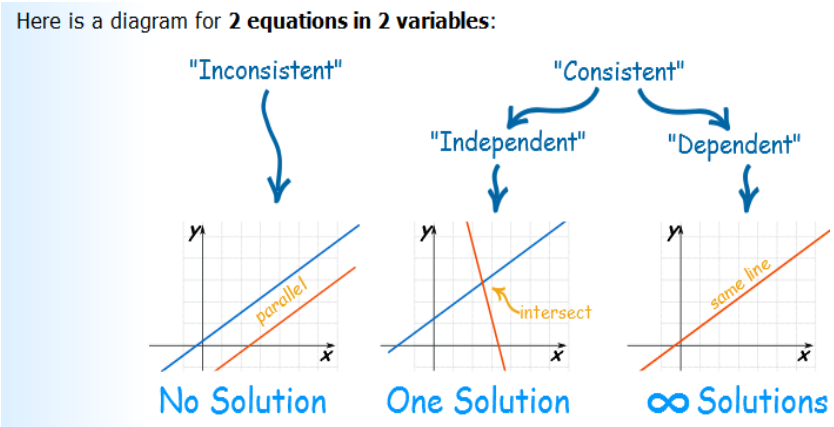
	dependent.
2. Consistency A linear system is **inconsistent** if it has no solution, and otherwise it is said to be **consistent**. When the system is inconsistent, it is possible to derive a contradiction from the equations, that may always be rewritten as the statement 0 = 1. For example, the equations $3x + 2y = 6$ and $3x + 2y = 12$ are inconsistent.	 The equations $3x + 2y = 6$ and $3x + 2y = 12$ are inconsistent
3. Equivalence Two linear systems using the same set of variables are equivalent if each of the equations in the second system can be derived algebraically from the equations in the first system, and vice versa.	

Solving a linear equation: When the number of equations is the **same** as the number of variables, there is **likely** to be a solution. Not guaranteed, but likely. There are only three possible cases:

- **No** solution
- **One** solution
- **Infinitely many** solutions

When there is **no solution** the equations are called **"inconsistent"**. **One** or **infinitely many solutions** are called **"consistent".**

Here is a diagram for **2 equations in 2 variables**:



"Independent" means that each equation gives new information. Otherwise they are **"Dependent"**. Also called "Linear Independence" and "Linear Dependence".

Methods of solving systems of linear equations:

The different methods of solving simultaneous linear equations can be broadly grouped as,

1. Graphical solution
2. Algebraic solution
   - Describing the solution
   - Evaluation of determinants:
   - **Direct method**
     - Example: Solving by elimination – guass elimination, guass Jordan elimination.
   - **Iterative method**
     - Example : Solving by substitution
   - Matrix solution:
   - Using computers
   - Pivoting
   - LU Decomposition etc.
3. Manual

1. Graphical solution:

The **graphical solution** of linear simultaneous equations is the point of intersection found by drawing the two linear equations on the same axes.

Example 1

Solve the following simultaneous equations graphically.

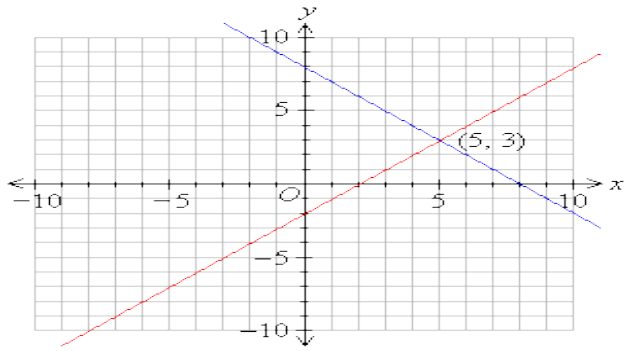
$x+y=8$

$x-y=2$

Solution:

The graphical solution of the simultaneous equations $x+y=8$ and $x-y=2$ is given by the point of intersection of the linear equations.

Consider $x + y = 8$. x-intercept: When $y = 0$, $x = 8$ y-intercept: When $x = 0$, y = 8	Consider $x – y = 2$. x-intercept: When $y = 0$, $x = 2$ y-intercept: When $x = 0$, $-y = 2$ $\therefore y = -2$



The diagram shows that the lines intersect at the point (5, 3). So, the solution of the simultaneous equations is $x = 5$ and $y = 3$ or (5, 3).

Note: Often the answer obtained with the graphical method is not exact.

2. Describing the solution

When the solution set is finite, it is reduced to a single element. In this case, the unique solution is described by a sequence of equations whose left-hand sides are the names of the unknowns and right-hand sides are the corresponding values, for example $(x = 3,\ y = -2,\ z = 6)$ When an order on the unknowns has been fixed, for example the alphabetical order the solution may be described as a vector of values, like $(3,\ -2,\ 6)$. It can be difficult to describe a set with infinite solutions. Typically, some of the variables are designated as **free** (or **independent**, or as **parameters**), meaning that they are allowed to take any value, while the remaining variables are **dependent** on the values of the free variables.

3. **Evaluation of determinants:** This is satisfactory for small number of simultaneous equations. Evaluation of determinants also called carmer's rule.

4. Direct method: the direct methods solve the problem by a finite seqyence of operations. In absence of rounding errors, direct method would deliver an exact solution.

Example: **Elimination of variables – guassian elimination (row reduction), guassian Jordan etc**

The direct method is based on elimination of variables to transform the set of equations to a triangular form.

Elimination: "Eliminate" means to **remove**: this method works by removing variables until there is just one left. The simplest method for solving a system of linear equations is to repeatedly eliminate variables. This method can be described as follows:

1. In the first equation, solve for one of the variables in terms of the others.
2. Substitute this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.
3. Repeat until the system is reduced to a single linear equation.

4. Solve this equation and then back-substitute until the entire solution is found.

Guass elimination: This method for solving a pair of simultaneous linear equations reduces one equation to one that has only a single variable. Once this has been done, the solution is the same as that for when one line was vertical or parallel. This method is known as the ***Gaussian elimination method***. This method is sometimes referred as elimination by adding.

Steps: Gaussian elimination is summarized by the following three steps:

1. Write the system of equations in matrix form. Form the augmented matrix. You omit the symbols for the variables, the equal signs, and just write the coefficients and the unknowns in a matrix. You should consider the matrix as shorthand for the original set of equations.
2. Perform elementary row operations to get zeros below the diagonal.
3. An elementary row operation is one of the following:
   - multiply each element of the row by a non-zero constant
   - Switch two rows
   - add (or subtract) a non-zero constant times a row to another row
4. Inspect the resulting matrix and re-interpret it as a system of equations.

If you get 0 = a non-zero quantity then there is no solution.

- If you get less equations than unknowns after discarding equations of the form 0=0 and if there isa solution then there is an infinite number of solutions
- If you get as many equations as unknowns after discarding equations of the form 0=0 and if there is a solution then there is exactly one solution

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Examples:

$$\begin{aligned} x+y+z &= 6 \\ 2x-y+z &= 3 \\ x+z &= 4 \end{aligned}$$

First form the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

—

Next add -2 times the first row to the second row and then add -1 times the first row to the third row:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & 0 & -2 \end{pmatrix}$$

Next multiply the second row by -1 and the third row by -1, just to get rid of the minus signs. Then switch the second and third rows:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 9 \end{pmatrix}$$

Now add -3 times the second row to the third row, so we have all zeros below the diagonal:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Now re-interpret the augmented matrix as a system of equations, starting at the bottom and working backwards (this is called back substitution).

The bottom equation is $0x + 0y + z = 3$ so $z = 3$.

The next to the bottom equation is $0x + y + 0z = 2$ so $y = 2$.

The next equation (the top one) is $x + y + z = 6$. Substitute the values $z = 3$ and $y = 2$ into the equation and get $x = 1$.

Guass Jordan Elimination: The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form described below, which is called the reduced row echelon form (RREF).
(a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
(b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
(c) Each column that contains a leading 1 has zeros in all other entries.
(d) The leading 1 in any row is to the left of any leading 1's in the rows below it.
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.
Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

5. Iterative method: it is a successive approximation procedure.

Types:

1. Stationary iterative methods
    a) Jacobi method
    b) Gauss seidel method
    c) Successive over relaxation method

2. Krylov subspace methods

a) conjurate gradient method (CG)
b) generalized minimal residual method (GMBES)
c) Bi_Conjurate gradient method.

Gauss seidel method: Given a general set of equations and unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with on the left hand side, the second equation is rewritten with on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$
$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$
$$\vdots$$
$$\vdots$$
$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$
$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

.
.
.

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\epsilon_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

where x_i^{new} is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.
When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example:

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations given in Example 1.

The first computation is identical to that given in Example 1. That is, using $(x_1, x_2, x_3) = (0, 0, 0)$ as the initial approximation, you obtain the following new value for x_1.

$$x_1 = -\frac{1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

Now that you have a new value for x_1, however, use it to compute a new value for x_2. That is,

$$x_2 = \frac{2}{9} + \frac{3}{9}(-0.200) - \frac{1}{9}(0) \approx 0.156.$$

Similarly, use $x_1 = -0.200$ and $x_2 = 0.156$ to compute a new value for x_3. That is,

$$x_3 = -\frac{3}{7} + \frac{2}{7}(-0.200) - \frac{1}{7}(0.156) \approx -0.508.$$

So the first approximation is $x_1 = -0.200$, $x_2 = 0.156$, and $x_3 = -0.508$. Continued iterations produce the sequence of approximations shown in Table 10.2.

TABLE 10.2

n	0	1	2	3	4	5
x_1	0.000	−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

6. Other methods:

- Matrix solution:
- Using computers
- Manual
- Pivoting
- LU Decomposition etc.

Pivoting:

Pivot element is the element of a matrix or an array, which is selected first by an algorithm (ex: Gaussian elimination, simplex algorithm etc), to do certain calculations. In the case of matrix algorithms, a pivot entry is usually required to be at least distinct from zero, and often distant from it, in this case finding this element is called pivoting.

Pivoting may be followed by an interchange of rows or columns to bring the pivot to a fixe position and allow the algorithm to proceed successfully and to reduce the round off error.

Pivoting may be thought of as swapping or sorting rows or column in a matrix and thus can be represented as multiplication by permutation matrices.

Over all pivoting adds more operations to the computational cost of an algorithm.

LU Decomposition method (LU Factorization):

A square matrix is said to be **lower triangular** matrix if the entries above the principle diagonal is zero.

A square matrix is said to be **upper triangular** matrix if the entries below the principle diagonal is zero.

A lower triangular matrix is said **to unit lower triangular matrix** if the principle diagonal of lower triangle matrix is equal to 1.

In numerical analysis and linear algebra, LU decomposition (where 'LU' stands for '**lower upper**', and also called LU factorization) factors a matrix as the product of a lower triangular matrix and an upper triangular matrix. The product sometimes includes a permutation matrix as well. The LU decomposition can be viewed as the matrix form of Gaussian elimination.

Definition: Let *A* be a square matrix. An **LU factorization** refers to the factorization of A, with proper row and/or column orderings or permutations, into two factors, a lower triangular matrix *L* and an upper triangular matrix *U*,

A=LU

In the lower triangular matrix all elements above the diagonal are zero, in the upper triangular matrix, all the elements below the diagonal are zero. For example, for a 3-by-3 matrix A, its LU decomposition looks as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Without a proper ordering or permutations in the matrix, the factorization may fail to materialize. For example, it is easy to verify (by expanding the matrix multiplication) that $a_{11} = l_{11}u_{11}$. If $a_{11} = 0$, then at least one of l_{11} and u_{11} has to be zero, which implies either *L* or *U* is singular. This is impossible if *A* is nonsingular. This is a procedural problem. It can be removed by simply reordering the rows of *A* so that the first element of the permuted matrix is nonzero. The same problem in subsequent factorization steps can be removed the same way; see the basic procedure below.

Steps:

Consider system of equations:

$$\begin{pmatrix} a_{11x} + a_{12y} + a_{13z} = b_1 \\ a_{21x} + a_{22y} + a_{23z} = b_2 \\ a_{31x} + a_{32y} + a_{33z} = b_3 \end{pmatrix}$$

This equation may be written as AX = b

$$A = \begin{pmatrix} a_{11x} + a_{12y} + a_{13z} = b_1 \\ a_{21x} + a_{22y} + a_{23z} = b_2 \\ a_{31x} + a_{32y} + a_{33z} = b_3 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} b1 \\ b2 \\ b3 \end{pmatrix}$$

Let A be a non singular matrix and let A=LU where L=unit lower triangle matrix and U=upper triangle matrix i.e.,

$$L = \begin{matrix} 1 & 0 & 0 \\ L21 & 1 & 0 \\ L31 & L32 & 1 \end{matrix} \qquad U = \begin{matrix} U11 & U12 & U13 \\ 0 & U22 & U23 \\ 0 & 0 & U33 \end{matrix}$$

We have system or equation AX=B.

W.K.T A=LU therefore LUX = B -------> 1

Put UX = Y where

$$Y = \begin{matrix} Y1 \\ Y2 \\ y3 \end{matrix}$$

are some unknowns

Therefore equation 1 becomes LY = B and we have UX = Y. first solve LY = B for Y using forward substitution method and then solve UX =Y for X using backward substitution.

Difference between direct and iterative methods:

"Direct" techniques use a "formula", whereas "indirect" techniques iterate until convergence. The criteria considered ***are time to converge, number of iterations, memory requirements and accuracy.*** In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

In direct method, one has to do a certain amount of work, till the end to be close to the exact solution. And if the working is incomplete, then, the solution is approximate.

In iterative method you can decide how much work you want to invest depending on how accurate you need your solution.

The second big difference is that, for a direct method, you generally need to have the entire matrix stored in memory. Not so in iterative methods: here, you typically only need a way to compute the application of the matrix to a vector, i.e., the matrix-vector product, So if you have a very large matrix, but you can relatively quickly compute its application to a vector (for instance, because the matrix is very sparse or has some other kind of structure), you can use this to your advantage.

Ill conditioned equations:

For any system of linear equations, the question of how many errors are there in a solution obtained by a numerical method is not readily answered.

1) Errors in the co-efficient and constants: in many practical cases, the co-efficient of the variables and also the constants on the RHS of the equation are obtained from observations of experiments or from other numerical calculations. They will have errors and hence the solution will too contain error.
2) Round off errors and number of operations:- numerical methods for solving systems of linear equations involve large number of arithmetic operations. Since round off errors are propagated at each step of an algorithm, the growth of round off errors can be such that, when n is large, a solution differs greatly from the true one.

ILL conditioning:

Certain systems of linear equations are such that their solution are very sensitive to small changes and therefore to errors, in their coefficients and constants. Such system are said to be ill conditioned.

If a system is ill conditioned, a solution obtained by a numerical method may be differ greatly from exact solution, even through great case is taken to keep round off and other errors very small.

Example: consider $2x + y = 4$

$$2x + 1.01y = 4.02$$

The exact solution is x=1, y=2. Suppose the coefficients of 2nd equation is changed by 1% and the constant of the 1st equation by 5%, this yields $2x + y = 3.8$ and $2.02\ x + y = 4.02$

Note: partial pivoting helps in reducing of errors.