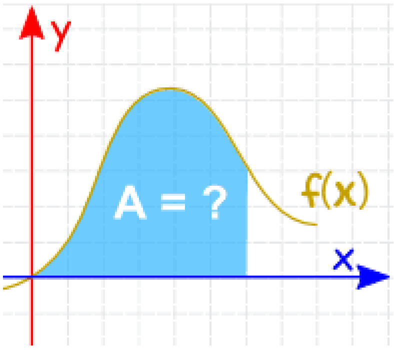
UNIT-2
Numerical Integration: Simpson's 1/3 and 3/8 rule, Trapezoidal rule.

Introduction to Integration

Integration is a way of adding slices to find the whole. The process of finding a function, given its derivative is called anti-differentiation or integration. If f`(x)=f(x), we say f(x) is an anti-derivative of f``(x).

Integration can be used to find areas, volumes, central points and many useful things. But it is easiest to start with finding the **area under the curve of a function** as follows:



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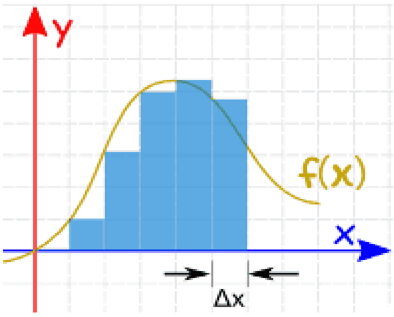
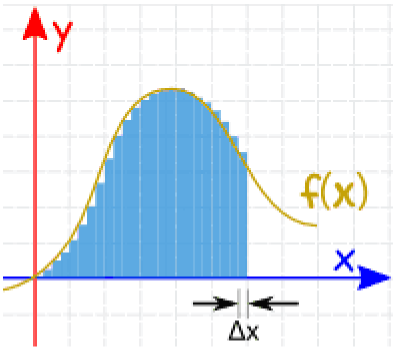


Slices or integral: An integral is a mathematical object that can be interpreted as an area or a generalization of area. Integrals, together with derivatives, are the fundamental objects of calculus. Other words for integral include anti-derivative and primitive.

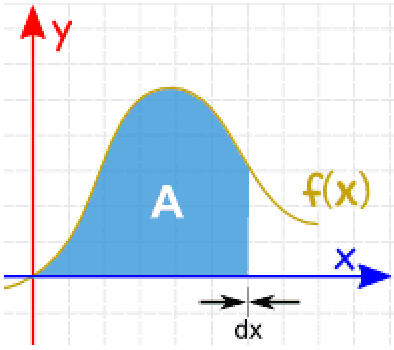
We could calculate the function at a few points and **add up slices of width Δx** like this (but the answer won't be very accurate):



We can make **Δx** a lot smaller and **add up many small slices** (answer is getting better):

And as the slices **approach zero in width**, the answer approaches the **true answer**.

We now write **dx** to mean the **Δx** slices are approaching zero in width.



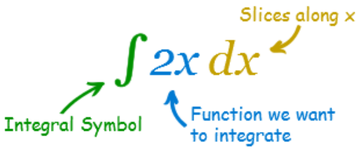
Note: finding an Integral is the reverse **of finding a Derivative.**

Example: integral of 2x

integral of 2x is x^2 , as the derivative of x^2 is 2x

Notation:

The symbol for "Integral" is a stylish "S" (for "Sum", the idea of summing slices):



After the Integral Symbol we put the function we want to find the integral of (called the Integrand), and then finish with **dx** to mean the slices go in the x direction (and approach zero in width).

And here is how we write the answer:

$$\int 2x \, dx = x^2 + C$$

Here C is the "Constant of Integration". It is there because of **all the functions whose derivative is 2x**: for example, The derivative of $\mathbf{x^2+4}$ is **2x**, and the derivative of $\mathbf{x^2+99}$ is also **2x**, and so on, Because the derivative of a constant is zero.

So when we **reverse** the operation (to find the integral) we only know **2x**, but there could have been a constant of any value. So we wrap up the idea by just writing + C at the end.

Rules of integration:

Examples			
Sl no	**Common Functions**	**Function**	**Integral**
1	Constant	$\int a \, dx$	ax + C
2	Variable	$\int x \, dx$	$x^2/2 + C$
3	Square	$\int x^2 \, dx$	$x^3/3 + C$

4	Reciprocal	$\int(1/x)\,dx$	$\ln\|x\| + C$
5	Exponential	$\int e^x\,dx$	$e^x + C$
		$\int a^x\,dx$	$a^x/\ln(a) + C$
		$\int \ln(x)\,dx$	$x\ln(x) - x + C$
6	Trigonometry (x in radians)	$\int \cos(x)\,dx$	$\sin(x) + C$
		$\int \sin(x)\,dx$	$-\cos(x) + C$
		$\int \sec^2(x)\,dx$	$\tan(x) + C$
	Rules	**Function**	**Integral**
1	Multiplication by constant	$\int cf(x)\,dx$	$c\int f(x)\,dx$
2	Power Rule ($n\neq-1$)	$\int x^n\,dx$	$x^{n+1}/(n+1) + C$
3	Sum Rule	$\int(f + g)\,dx$	$\int f\,dx + \int g\,dx$
4	Difference Rule	$\int(f - g)\,dx$	$\int f\,dx - \int g\,dx$
5	Integration by Parts		
6	Substitution Rule		

Integration by parts:

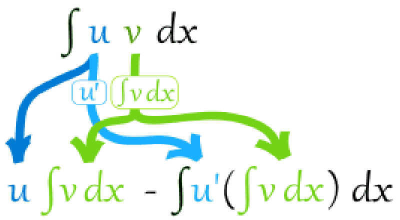
Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

You will see plenty of examples soon, but first let us see the rule:

$\int u\,v\,dx = u\int v\,dx - \int u'\,(\int v\,dx)\,dx$

- **u** is the function u(x)
- **v** is the function v(x)

As a diagram:



Integration by Substitution

"Integration by Substitution" (also called "u-substitution") is a method to find an integral, but only when it can be set up in a special way.

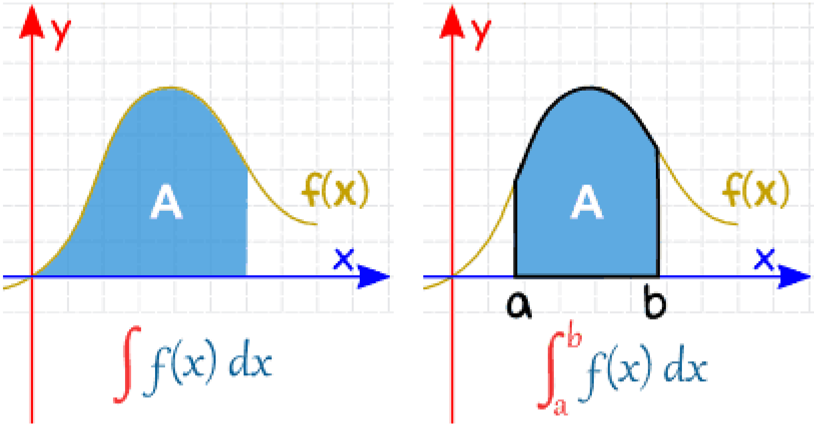
The first and most vital step is to be able to write our integral in this form:

$$\int f(g(x))\, g'(x)\, dx$$

Note that we have **g(x)** and its derivative **g'(x)**

Definite and Indefinite Integrals

Definite Integral has actual values to calculate between (they are put at the bottom and top of the "S"):



Indefinite Integral **Definite** Integral

A **Definite Integral** has start and end values: in other words there is an **interval** (a to b).

Analytical integrations or symbolic are exact and obtained by methods of symbolic manipulation, derived using analysis.

Numerical analysis usually indicates an approximate solution obtained by methods of numerical analysis.

Numerical integration: In numerical analysis, numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral and by extension, the term is also sometimes used to describe the numerical solution of differential equations.

Numerical integration is the approximate computation of an integral using numerical techniques, it is also called quadrature.

Methods: the choice of methods depends in part on the results required and in part on the function or data to be integrated

- Manual method
- Trapezium rule
- Midpoint rule
- Simpson’s rule
- Quadratic triangulation
- Romberg integration

- Guass quadrature.

Applications of integration: integration is used to find

1. Area under a curve
2. Area between 2 curves
3. Volume of solid of revolution
4. Centroid of an area
5. Moments of inertia
6. Work by a variable force
7. Displacement, velocity etc

Simpson's Rule Formula

Simpson's Rule Formula is used to calculate the integral value of any function. It calculates the value of the area under any curve over a given interval by dividing the area into equal parts. It follows the method similar to integration by parts.

In order to integrate any function f(x) in the interval (a,b), follow the steps given below:

1. Select a value for n, which is the number of parts the interval is divided into. Let the value of n be an even number.
2. Calculate the width, h = $\frac{b-a}{n}$.
3. Calculate the values of x_0 to x_n as $x_0 = a$, $x_1 = x_0 + h$,.....$x_{n-1} = x_{n-2} + h$, $x_n = b$.
4. Consider y = f(x). Now find the values of y(y_0 to y_n) for the corresponding x(x_0 to x_n) values.
5. Substitute all the above found values in the Simpson's Rule Formula to calculate the integral value.

Simpson's One Third Rule Formula is

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

Simpson 1/3 rule Algorithm:

1. **Start**
2. Given a function f(x)
3. Read a, b, n (Get user inputs – a and b end points of interval), n is the number of intervals (even)
4. Do the integration
5. Set h=(b-a)/n
6. Set sum = 0
7. Begin for i=1 to n-1
   Set x= a+h*i
   If i%2=0
   Then set sum=sum+2*f(x)
   Else

Set sum=sum+4*f(x)
End for
8. Set sum=sum+f(a)+f(b)
9. Set ans = sum*(h/3)

Simpson 3/8 rule formula:

$$\int_a^b f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + ... + y_{n-1}) + 2(y_3 + y_6 + ... + y_{n-3})]$$

Simpson 3/8 rule Algorithm:

1. **Start**
2. Given a function f(x)
3. Read a, b, n (Get user inputs – a and b end points of interval), n is the number of intervals (even)
4. Do the integration
5. Set h=(b-a)/n
6. Set sum = 0
7. Begin for i=1 to n-1
   Set x= a+h*i
   If i%3=0
   Then set sum=sum+2*f(x)
   Else
   Set sum=sum+3*f(x)
   End for
8. Set sum=sum+f(a)+f(b)
9. Set ans = sum*(3h/8)

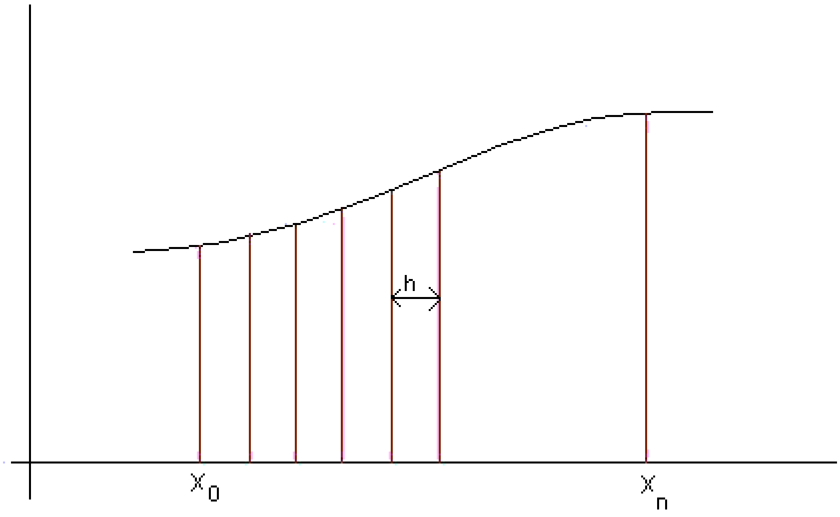
Tapezoidal rule:

Definite integral is replica of area under a curve within the certain limit. In order to calculate such area, there have been developed a number of analytical methods but they **are time-consuming, laborious and chance of occurrence of error is also high**. That is why, techniques of numerical methods are very much popular in calculation numerical integration which can easily be programmed and trapezoidal method is one of them.

Trapezoidal method is based on the principle that the area under the curve which is to be calculated is divided into number of small segments. The bounding curve in the segment is considered to be a straight line as a result the small enclosed area becomes a trapezium.

The area of each small trapezium is calculated and summed up i.e. integrated. This idea is the working mechanism in trapezoidal method algorithm and flowchart, even it source code.

Let us consider a function f(x) representing a curve as shown in above figure. You are to find out the area under the curve from point ‘a’ to ‘b’. In order to do so, divide the distance between ab into a number vertical strips of width ‘h’ so that each strip can be considered as trapezium.



The following formula is used to calculate the area under the curve:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{1}{2}h[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

Trapezoidal Method Algorithm:

- Start
- Define and Declare function
- Input initial boundary value, final boundary value and length of interval
- Calculate number of strips, n = (final boundary value –final boundary value)/length of interval
- Perform following operation in loop

x[i]=x0+i*h

y[i]=f(x[i])

print y[i]

- Initialize se=0, s0=0
- Do the following using loop

If i %2 = 0

So=s0+y[i]

Otherwise

Se=se+y[i]

- ans= h/3*(y[0]+y[n]+4*so+2*se)
- print the ans
- stop